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# ON SIGN-SYMMETRIC SIGNED GRAPHS

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**ABSTRACT.** A signed graph is said to be sign-symmetric if it is switching isomorphic to its negation. Bipartite signed graphs are trivially sign-symmetric. We give new constructions of non-bipartite sign-symmetric signed graphs. Sign-symmetric signed graphs have a symmetric spectrum but not the other way around. We present constructions of signed graphs with symmetric spectra which are not sign-symmetric. This, in particular, answers a problem posed by Belardo, Cioabă, Koolen, and Wang (2018).

## 1. INTRODUCTION

Let  $G$  be a graph with vertex set  $V$  and edge set  $E$ . All graphs considered in this paper are undirected, finite, and simple (without loops or multiple edges).

A *signed graph* is a graph in which every edge has been declared positive or negative. In fact, a signed graph  $\Gamma$  is a pair  $(G, \sigma)$ , where  $G = (V, E)$  is a graph, called the underlying graph, and  $\sigma : E \rightarrow \{-1, +1\}$  is the sign function or signature. Often, we write  $\Gamma = (G, \sigma)$  to mean that the underlying graph is  $G$ . The signed graph  $(G, -\sigma) = -\Gamma$  is called the *negation* of  $\Gamma$ . Note that if we consider a signed graph with all edges positive, we obtain an unsigned graph.

Let  $v$  be a vertex of a signed graph  $\Gamma$ . *Switching* at  $v$  is changing the signature of each edge incident with  $v$  to the opposite one. Let  $X \subseteq V$ . Switching a vertex set  $X$  means reversing the signs of all edges between  $X$  and its complement. Switching a set  $X$  has the same effect as switching all the vertices in  $X$ , one after another.

Two signed graphs  $\Gamma = (G, \sigma)$  and  $\Gamma' = (G, \sigma')$  are said to be *switching equivalent* if there is a series of switching that transforms  $\Gamma$  into  $\Gamma'$ . If  $\Gamma'$  is isomorphic to a switching of  $\Gamma$ , we say that  $\Gamma$  and  $\Gamma'$  are *switching isomorphic* and we write  $\Gamma \simeq \Gamma'$ . The signed graph  $-\Gamma$  is obtained from  $\Gamma$  by reversing the sign of all edges. A signed graph  $\Gamma = (G, \sigma)$  is said to be *sign-symmetric* if  $\Gamma$  is switching isomorphic to  $(G, -\sigma)$ , that is:  $\Gamma \simeq -\Gamma$ .

For a signed graph  $\Gamma = (G, \sigma)$ , the adjacency matrix  $A = A(\Gamma) = (a_{ij})$  is an  $n \times n$  matrix in which  $a_{ij} = \sigma(v_i v_j)$  if  $v_i$  and  $v_j$  are adjacent, and 0 if they are not. Thus  $A$  is a symmetric matrix with entries  $0, \pm 1$  and zero diagonal, and conversely, any such matrix is the adjacency matrix of a signed graph. The spectrum of  $\Gamma$  is the list of eigenvalues of its adjacency matrix with their multiplicities. We say that  $\Gamma$  has a

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*symmetric spectrum* (with respect to the origin) if for each eigenvalue  $\lambda$  of  $\Gamma$ ,  $-\lambda$  is also an eigenvalue of  $\Gamma$  with the same multiplicity.

Recall that (see [4]), the *Seidel adjacency matrix* of a graph  $G$  with the adjacency matrix  $A$  is the matrix  $S$  defined by

$$S_{uv} = \begin{cases} 0 & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 1 & \text{if } u \not\sim v \end{cases}$$

so that  $S = J - I - 2A$ . The Seidel adjacency spectrum of a graph is the spectrum of its Seidel adjacency matrix. If  $G$  is a graph of order  $n$ , then the Seidel matrix of  $G$  is the adjacency matrix of a signed complete graph  $\Gamma$  of order  $n$  where the edges of  $G$  are precisely the negative edges of  $\Gamma$ .

**Proposition 1.1.** *Suppose  $S$  is a Seidel adjacency matrix of order  $n$ . If  $n$  is even, then  $S$  is nonsingular, and if  $n$  is odd,  $\text{rank}(S) \geq n - 1$ . In particular, if  $n$  is odd, and  $S$  has a symmetric spectrum, then  $S$  has an eigenvalue 0 of multiplicity 1.*

*Proof.* We have  $\det(S) \equiv \det(I - J) \pmod{2}$ , and  $\det(I - J) = 1 - n$ . Hence, if  $n$  is even,  $\det(S)$  is odd. So,  $S$  is nonsingular. Now, if  $n$  is odd, any principal submatrix of order  $n - 1$  is nonsingular. Therefore,  $\text{rank}(S) \geq n - 1$ .  $\square$

The goal of this paper is to study sign-symmetric signed graphs as well as signed graphs with symmetric spectra. It is known that bipartite signed graphs are sign-symmetric. We give new constructions of non-bipartite sign-symmetric graphs. It is obvious that sign-symmetric graphs have a symmetric spectrum but not the other way around (see Remark 4.1 below). We present constructions of graphs with symmetric spectra which are not sign-symmetric. This, in particular answers a problem posed in [2].

## 2. CONSTRUCTIONS OF SIGN-SYMMETRIC GRAPHS

We note that the property that two signed graphs  $\Gamma$  and  $\Gamma'$  are switching isomorphic is equivalent to the existence of a ‘signed’ permutation matrix  $P$  such that  $PA(\Gamma)P^{-1} = A(\Gamma')$ . If  $\Gamma$  is a bipartite signed graph, then we may write its adjacency matrix as

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}.$$

It follows that  $PAP^{-1} = -A$  for

$$P = \begin{bmatrix} -I & O \\ O & I \end{bmatrix},$$

which means that bipartite graphs are ‘trivially’ sign-symmetric. So it is natural to look for non-bipartite sign-symmetric graphs. The first construction was given in [1] as follows.

**Theorem 2.1.** *Let  $n$  be an even positive integer and  $V_1$  and  $V_2$  be two disjoint sets of size  $n/2$ . Let  $G$  be an arbitrary graph with the vertex set  $V_1$ . Construct the complement of  $G$ , that is  $G^c$ , with the vertex set  $V_2$ . Assume that  $\Gamma = (K_n, \sigma)$  is a signed complete graph in which  $E(G) \cup E(G^c)$  is the set of negative edges. Then the spectrum of  $\Gamma$  is sign-symmetric.*

Theorem 2.1 says that for an even positive integer  $n$ , let  $B$  be the adjacency matrix of an arbitrary graph on  $n/2$  vertices. Then, the complete signed graph in which the negatives edges induce the disjoint union of  $G$  and its complement, is sign-symmetric.

**2.1. Constructions for general signed graphs.** Let  $\mathcal{M}_{r,s}$  denote the set of  $r \times s$  matrices with entries from  $\{-1, 0, 1\}$ . We give another construction generalizing the one given in Theorem 2.1:

**Theorem 2.2.** *Let  $B, C \in \mathcal{M}_{k,k}$  be symmetric matrices where  $B$  has a zero diagonal. Then the signed graph with the adjacency matrices*

$$A = \begin{bmatrix} B & C \\ C & -B \end{bmatrix}$$

*is sign-symmetric on  $2k$  vertices.*

*Proof.*

$$\begin{bmatrix} O & -I \\ I & O \end{bmatrix} \begin{bmatrix} B & C \\ C & -B \end{bmatrix} \begin{bmatrix} O & I \\ -I & O \end{bmatrix} = \begin{bmatrix} -B & -C \\ -C & B \end{bmatrix} = -A$$

□

Note that Theorem 2.2 shows that there exists a sign-symmetric graph for every even order.

We define the family  $\mathcal{F}$  of signed graphs as those which have an adjacency matrix satisfying the conditions given in Theorem 2.2. To get an impression on what the role of  $\mathcal{F}$  is in the family of sign-symmetric graphs, we investigate small complete signed graphs. All but one complete signed graphs with symmetric spectra of orders 4, 6, 8 are illustrated in Fig. 6 (we show one signed graph in the switching class of the signed complete graphs induced by the negative edges). There is only one sign-symmetric complete signed graph of order 4. There are four complete signed graphs with symmetric spectrum of order 6, all of which are sign-symmetric, and twenty-one complete signed graphs with symmetric spectrum of order 8, all except the last one are sign-symmetric, and together with the negation of the last signed graph, Fig. 6 gives all complete signed graphs with symmetric spectrum of order 4, 6 and 8. Interestingly, all of the above sign-symmetric signed graphs belong to  $\mathcal{F}$ .

The following proposition shows that  $\mathcal{F}$  is closed under switching.

**Proposition 2.3.** *If  $\Gamma \in \mathcal{F}$  and  $\Gamma'$  is obtained from  $\Gamma$  by switching, then  $\Gamma' \in \mathcal{F}$ .*

*Proof.* Let  $\Gamma \in \mathcal{F}$ . It is enough to show that if  $\Gamma'$  is obtained from  $\Gamma$  by switching with respect to its first vertex, then  $\Gamma' \in \mathcal{F}$ . We may write the adjacency matrix of

$\Gamma$  as follows:

$$A = \left[ \begin{array}{c|c|c|c} 0 & \mathbf{b}^\top & c & \mathbf{c}^\top \\ \hline \mathbf{b} & B' & \mathbf{c} & C' \\ \hline c & \mathbf{c}^\top & 0 & -\mathbf{b}^\top \\ \hline \mathbf{c} & C' & -\mathbf{b} & -B' \end{array} \right].$$

After switching with respect to the first vertex of  $\Gamma$ , the adjacency matrix of the resulting signed graph is

$$\left[ \begin{array}{c|c|c|c} 0 & -\mathbf{b}^\top & -c & -\mathbf{c}^\top \\ \hline -\mathbf{b} & B' & \mathbf{c} & C' \\ \hline -c & \mathbf{c}^\top & 0 & -\mathbf{b}^\top \\ \hline -\mathbf{c} & C' & -\mathbf{b} & -B' \end{array} \right].$$

Now by interchange the 1st and  $(k+1)$ -th rows and columns we obtain

$$\begin{bmatrix} 0 & \mathbf{c}^\top & -c & -\mathbf{b}^\top \\ \mathbf{c} & B' & -\mathbf{b} & C' \\ -c & -\mathbf{b}^\top & 0 & -\mathbf{c}^\top \\ -\mathbf{b} & C' & -\mathbf{c} & -B' \end{bmatrix}$$

which is a matrix of the form given in Theorem 2.2 and thus  $\Gamma'$  is isomorphic with a signed graph in  $\mathcal{F}$ .  $\square$

In the following we present two constructions for complete sign-symmetric signed graphs using self-complementary graphs.

**2.2. Constructions for complete signed graphs.** In the following, the meaning of a self-complementary graph is the same as defined for unsigned graphs. Let  $G$  be a self-complementary graph so that there is a permutation matrix  $P$  such that  $PA(G)P^{-1} = A(\overline{G})$  and  $PA(\overline{G})P^{-1} = A(G)$ . It follows that if  $\Gamma$  is a complete signed graph with  $E(G)$  being its negative edges, then  $A(\Gamma) = A(\overline{G}) - A(G)$ , (in other words,  $A(\Gamma)$  is the Seidel matrix of  $G$ ). It follows that  $PA(\Gamma)P^{-1} = -A(\Gamma)$ . So we obtain the following:

**Observation 2.4.** If  $\Gamma$  is a complete signed graph whose negative edges induce a self-complementary graph, then  $\Gamma$  is sign-symmetric.

We give one more construction of sign-symmetric signed graphs based on self-complementary graphs as a corollary to Observation 2.4. We remark that a self-complementary graph of order  $n$  exists whenever  $n \equiv 0$  or  $1 \pmod{4}$ .

**Proposition 2.5.** *Let  $G, H$  be two self-complementary graphs, and let  $\Gamma$  be a complete signed graph whose negative edges induce the join of  $G$  and  $H$  (or the disjoint union of  $G$  and  $H$ ). Then  $\Gamma$  is sign symmetric. In particular, if  $G$  has  $n$  vertices, and if  $H$  is a singleton, then the complete signed graph  $\Gamma$  of order  $n + 1$  with negative edges equal to  $E(G)$  is sign-symmetric.*

In the following remark we present a sign-symmetric construction for non-complete signed graphs.

*Remark 2.6.* Let  $\Gamma', \Gamma''$  be two signed graphs which are isomorphic to  $-\Gamma', -\Gamma''$ , respectively. Consider the signed graph  $\Gamma$  obtained from joining  $\Gamma'$  and  $\Gamma''$  whose negative edges are the union of negative edges in  $\Gamma'$  and  $\Gamma''$ . Then,  $\Gamma$  is sign-symmetric.

*Remark 2.7.* By Proposition 2.5, we have a construction of sign-symmetric complete signed graphs of order  $n \equiv 0, 1$  or  $2 \pmod{4}$ . All complete sign-symmetric signed graphs of order 5 and 9 (depicted in Fig. 7) can be obtained in this way. There is just one sign-symmetric signed graph of order 5 which is obtained by joining a vertex to a complete signed graph of order 4 whose negative edges form a path of length 3 (which is self-complementary). Moreover, there exist sixteen complete signed graphs of order 9 with symmetric spectrum of which ten are sign-symmetric; the first three are not sign-symmetric, and when we include their negations we get them all. All of these ten complete sign-symmetric signed graphs can be obtained by joining a vertex to a complete signed graph of order 8 whose negative edges induce a self-complementary graph. Note that there are exactly ten self-complementary graphs of order 8.

**Theorem 2.8.** *There exists a complete sign-symmetric signed graph of order  $n$  if and only if  $n \equiv 0, 1$  or  $2 \pmod{4}$ .*

*Proof.* Using the previous results obviously one can construct a sign-symmetric signed graph of order  $n$  whenever  $n \equiv 0, 1$  or  $2 \pmod{4}$ . Now, suppose that there is a complete sign-symmetric signed graph  $\Gamma$  of order  $n$  with  $n \equiv 3 \pmod{4}$ . By [7, Corollary 3.6], the determinant of the Seidel matrix of  $\Gamma$  is congruent to  $1 - n \pmod{4}$ . Since  $n \equiv 3 \pmod{4}$ , the determinant of the Seidel matrix (obtained from the negative edges of  $\Gamma$ ) is not zero. Hence, we can conclude that all eigenvalues of  $\Gamma$  are non-zero. Therefore,  $\Gamma$  cannot have a symmetric spectrum, and also it cannot be sign-symmetric.  $\square$

In [9] all switching classes of Seidel matrices of order at most seven are given. There is a error in the spectrum of one of the graphs on six vertices in [9, Table 4.1] (2.37 should be 2.24), except for that, the results in [9] coincide with ours.

### 3. POSITIVE AND NEGATIVE CYCLES

A graph whose connected components are  $K_2$  or cycles is called an *elementary graph*. Like unsigned graphs, the coefficients of the characteristic polynomial of the adjacency matrix of a signed graph  $\Gamma$  can be described in terms of elementary subgraphs of  $\Gamma$ .

**Theorem 3.1** ([3, Theorem 2.3]). *Let  $\Gamma = (G, \sigma)$  be a signed graph and*

$$(1) \quad P_\Gamma(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

*be the characteristic polynomial of the adjacency matrix of  $\Gamma$ . Then*

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{|c(B)|} \sigma(B),$$

*where  $\mathcal{B}_i$  is the set of elementary subgraphs of  $G$  on  $i$  vertices,  $p(B)$  is the number of components of  $B$ ,  $c(B)$  the set of cycles in  $B$ , and  $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$ .*

*Remark 3.2.* It is clear that  $\Gamma$  has a symmetric spectrum if and only if in its characteristic polynomial (1), we have  $a_{2k+1} = 0$ , for  $k = 1, 2, \dots$

In a signed graph, a cycle is called *positive* or *negative* if the product of the signs of its edges is positive or negative, respectively. We denote the number of positive and negative  $\ell$ -cycles by  $c_\ell^+$  and  $c_\ell^-$ , respectively.

**Observation 3.3.** For sign-symmetric signed graph, we have

$$c_{2k+1}^+ = c_{2k+1}^- \text{ for } k = 1, 2, \dots$$

*Remark 3.4.* If in a signed graph  $\Gamma$ ,  $c_{2k+1}^+ = c_{2k+1}^-$  for all  $k = 1, 2, \dots$ , then it is not necessary that  $\Gamma$  is sign-symmetric. See the complete signed graph given in Fig. 3. For this complete signed graph we have  $c_{2k+1}^+ = c_{2k+1}^-$  for all  $k = 1, 2, \dots$ , but it is not sign-symmetric. Moreover, one can find other examples among complete and non-complete signed graphs. For example, the signed graph given in Fig. 2 is a non-complete signed graph with the property that  $c_{2k+1}^+ = c_{2k+1}^-$  for all  $k = 1, 2, \dots$ , but it is not sign-symmetric.

By Theorem 3.1, we have that  $a_3 = 2(c_3^- - c_3^+)$ . By Theorem 3.1 and Remark 3.2 for signed graphs having symmetric spectrum, we have  $c_3^+ = c_3^-$ . Further, for each complete signed graph with a symmetric spectrum, it can be seen that  $c_5^+ = c_5^-$ . However, the equality  $c_{2k+1}^+ = c_{2k+1}^-$  does not necessarily hold for  $k \geq 3$ . The complete signed graph in Fig. 1 has a symmetric spectrum for which  $c_7^+ \neq c_7^-$ .

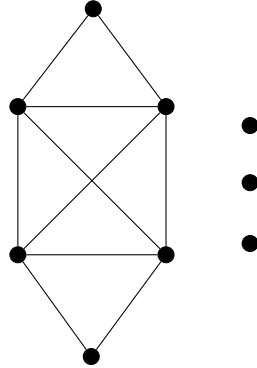


FIGURE 1. The graph induced by negative edges of a complete signed graph on 9 vertices with a symmetric spectrum but  $c_7^+ \neq c_7^-$

*Remark 3.5.* There are some examples showing that for a non-complete signed graph we have  $c_{2k+1}^+ = c_{2k+1}^-$  for all  $k = 1, 2, \dots$ , but their spectra are not symmetric. As an example see Fig. 2, (dashed edges are negative; solid edges are positive).

Now, we may ask a weaker version of the result mentioned in Remark 3.4 as follows.

**Question 3.6.** Is it true that if in a complete signed graph  $\Gamma$ ,  $c_{2k+1}^+ = c_{2k+1}^-$  for all  $k = 1, 2, \dots$ , then  $\Gamma$  has a symmetric spectrum?



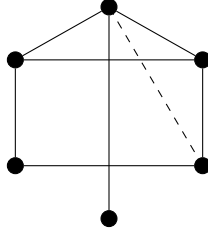


FIGURE 2. A signed graph with  $c_{2k+1}^+ = c_{2k+1}^-$  for  $k = 1, 2, \dots$ , but its spectrum is not symmetric

#### 4. SIGN-SYMMETRIC VS. SYMMETRIC SPECTRUM

*Remark 4.1.* Consider the complete signed graph whose negative edges induces the graph of Fig. 3. This graph has a symmetric spectrum, but it is not sign-symmetric. Note that this complete signed graph has the minimum order with this property. Moreover, for this complete signed graph we have the equalities  $c_{2k+1}^+ = c_{2k+1}^-$  for  $k = 1, 2, 3$ .

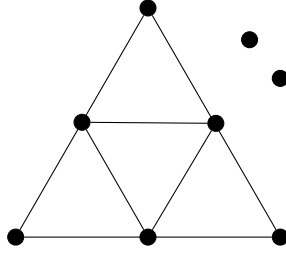


FIGURE 3. The graph induced by negative edges of a complete signed graph on 8 vertices with a symmetric spectrum but not sign-symmetric

*Remark 4.2.* A *conference matrix*  $C$  of order  $n$  is an  $n \times n$  matrix with zero diagonal and all off-diagonal entries  $\pm 1$ , which satisfies  $CC^T = (n-1)I$ . If  $C$  is symmetric, then  $C$  has eigenvalues  $\pm\sqrt{n-1}$ . Hence, its spectrum is symmetric. Conference matrices are well-studied; see for example [4, Section 10.4]. An important example of a symmetric conference matrix is the Seidel matrix of the Paley graph extended with an isolated vertex, where the *Paley graph* is defined on the elements of a finite field  $\mathbf{F}_q$ , with  $q \equiv 1 \pmod{4}$ , where two elements are adjacent whenever the difference is a nonzero square in  $\mathbf{F}_q$ . The Paley graph is self-complementary. Therefore, by Proposition 2.5,  $C$  is the adjacency matrix of a sign-symmetric complete signed graph. However, there exist many more symmetric conference matrices, including several that are not sign-symmetric (see [5]).

In [2], the authors posed the following problem on the existence of the non-complete signed graphs which are not sign-symmetric but have symmetric spectrum.

**Problem 4.3** ([2]). Are there non-complete connected signed graphs whose spectrum is symmetric with respect to the origin but they are not sign-symmetric?

We answer this problem by showing that there exists such a graph for any order  $n \geq 6$ . For  $s \geq 0$ , define the signed graph  $\Gamma_s$  to be the graph illustrated in Fig. 4.

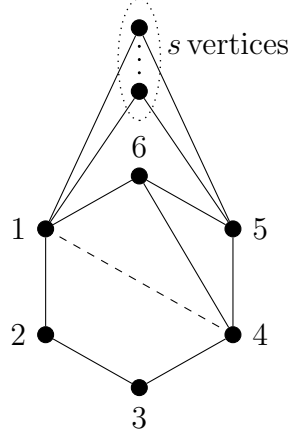


FIGURE 4. The graph  $\Gamma_s$

**Theorem 4.4.** *For  $s \geq 0$ , the graph  $\Gamma_s$  has a symmetric spectrum, but it is not sign-symmetric.*

*Proof.* Let  $S$  be the set of  $s$  vertices adjacent to both 1 and 5. The positive 5-cycles of  $\Gamma_s$  are 123461 together with  $u1645u$  for any  $u \in S$ , and the negative 5-cycles are  $u1465u$  for any  $u \in S$ . Hence,  $c_5^+ = s + 1$  and  $c_5^- = s$ . In view of Observation 3.3, this shows that  $\Gamma_s$  is not sign-symmetric.

Next, we show that  $\Gamma_s$  has a symmetric spectrum. It suffices to verify that  $a_{2k+1} = 0$  for  $k = 1, 2, \dots$

The graph  $\Gamma_s$  contains a unique positive cycle of length 3: 4564 and a unique negative cycle of length 3: 1461. It follows that  $a_3 = 0$ .

As discussed above, we have  $c_5^+ = s + 1$  and  $c_5^- = s$ . We count the number of positive and negative copies of  $K_2 \cup C_3$ . For the negative triangle 1461, there are  $s + 1$  non-incident edges, namely 23 and  $5u$  for any  $u \in S$  and for the positive triangle 4564, there are  $s + 2$  non-incident edges, namely 12, 23 and  $1u$  for any  $u \in S$ . It follows that

$$a_5 = -2((s + 1) - s) + 2((s + 2) - (s + 1)) = 0.$$

Now, we count the number of positive and negative elementary subgraphs on 7 vertices:

- $C_7$ :  $s$  positive:  $u123465u$  for any  $u \in S$ , and no negative;
- $K_2 \cup C_5$ :  $2s$  positive:  $u5 \cup 123461$ , and  $23 \cup u1645u$  for any  $u \in S$ , and  $s$  negative:  $23 \cup u1465u$  for any  $u \in S$ ;

$2K_2 \cup C_3$ :  $s+1$  positive:  $u1 \cup 23 \cup 4564$  for any  $u \in S$ , and  $s+1$  negative:  $u5 \cup 23 \cup 1461$  for any  $u \in S$ ;

$C_4 \cup C_3$ : none.

Therefore,

$$a_7 = -2(s-0) + 2(2s-s) - 2((s+1) - (s+1)) = 0.$$

The graph  $\Gamma_s$  contains no elementary subgraph on 8 vertices or more. The result now follows.  $\square$

More families of non-complete signed graphs with a symmetric spectrum but not sign-symmetric can be found. Consider the signed graphs  $\Gamma_{s,t}$  depicted in Fig. 5, in which the number of upper repeated pair of vertices is  $s \geq 0$  and the number of upper repeated pair of vertices is  $t \geq 1$ . In a similar fashion as in the proof of Theorem 4.4 it can be verified that  $\Gamma_{s,t}$  has a symmetric spectrum, but it is not sign-symmetric.

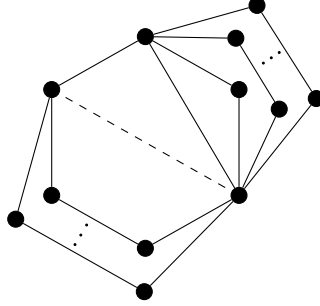


FIGURE 5. The family of signed graphs  $\Gamma_{s,t}$

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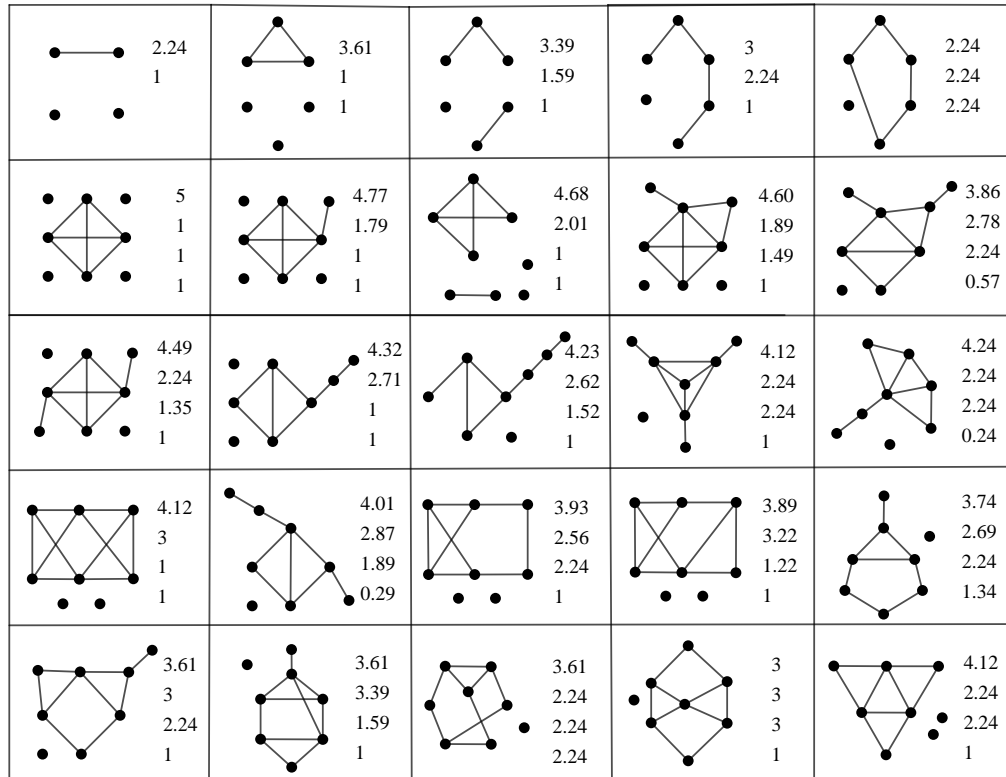


FIGURE 6. Complete signed graphs (up to switching isomorphism and negation) of order 4, 6, 8 having symmetric spectrum. The numbers next to the graphs are the non-negative eigenvalues. Only the last graph on the right is not sign-symmetric.

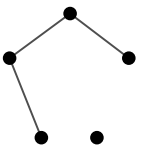
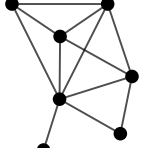
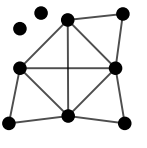
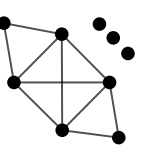
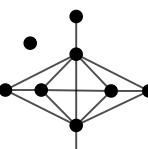
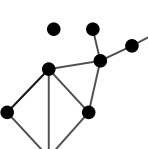
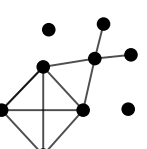
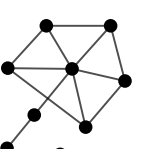
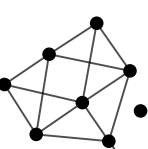
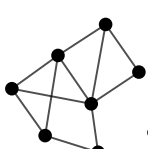
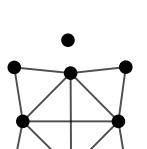

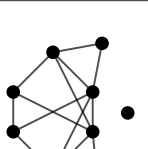
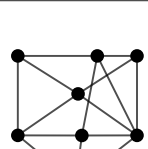
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 4.12 3 3 1 0	 4.24 3.61 2.24 0.24 0	 4.12 3 3 1 0	 4.12 4.12 1 1 0
 3.61 3.61 2.24 2.24 0	 3 3 3 3 0		

FIGURE 7. Complete signed graphs (up to switching isomorphism and negation) of order 5,9 having symmetric spectrum. The numbers next to the graphs are the non-negative eigenvalues. The first three signed graphs of order 9 are not sign-symmetric.